

$$\therefore q(t) = \frac{-i}{2} e^{it} + \frac{i}{2} e^{-t} + \left(\frac{-1+i}{2}\right) t e^{-t}, \quad t \gg 0 \rightarrow \textcircled{9}$$

For $F(t) = \cos t$ we use only the real part of $q(t)$ from eqn \textcircled{9} where $F(t)$ was $e^{it} = \cos t + i \sin t$.

$$\begin{aligned} \therefore q(t) &= \operatorname{Re} \left[\frac{-i}{2} e^{it} + \frac{i}{2} e^{-t} + \left(\frac{-1+i}{2}\right) t e^{-t} \right] \\ &= \frac{\sin t}{2} - \frac{t e^{-t}}{2} \quad \rightarrow \textcircled{10} \end{aligned}$$

$q_{\text{steady}}(t) = q_{\text{particular}}(t)$ is the soln. that survives in the case of large t . Since $t e^{-t} \rightarrow 0$ for $t \rightarrow \infty$, we notice that $q_{\text{steady}}(t) = \frac{1}{2} \sin t$ when $F(t) = \cos t$.

$$\left[\because \text{Phase difference} \right] = \frac{\pi}{2} \quad \left[\because \sin t = \cos(t - \frac{\pi}{2}) \right]$$

betw $F(t)$ & $q_{\text{steady}}(t)$

2. PROBLEM 21(c) HF Chapt. 3

Having done all the work in (b) itself, refer to (b) for soln.

$$q(t) = \frac{\sin t - t e^{-t}}{2}$$

$$\therefore \dot{q}(t) = \frac{1}{2} [\cos t - e^{-t} + t e^{-t}]$$

$$\text{For max. } q, \quad \dot{q}(t) = 0 \Rightarrow \cos t = (1-t) e^{-t} \rightarrow \textcircled{11}$$

Soln. to \textcircled{11} give pts. of max. response (though some soln. also correspond to min. response) \leftrightarrow

Redefining constants, $q(t) = Ae^{-t} +$

Plot of q v.s. time gives :

From graph, q_f is max. for $t \approx 1.7$

d) Let $\Theta(t)$ represent the step function at $t=0$

$$\therefore \Theta(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

\therefore As mentioned in problem, $\Theta(t) = S(t) \rightarrow \textcircled{2}$

Result of part (a) tells us that

$$q(t) = 1 - e^{-t}(1+t) \rightarrow \textcircled{3}$$

is a soln. to

$$\ddot{q}_f + 2\dot{q}_f + q_f = \Theta(t) \rightarrow \textcircled{4}$$

Differentiating eqn. (4), we get

$$\ddot{\tilde{q}}_f + 2\dot{\tilde{q}}_f + \tilde{q}_f = \dot{\Theta}$$

$$\therefore (\ddot{q}) + 2(\dot{q}) + (q) = S(t)$$

$$\therefore \ddot{\tilde{q}}_f + 2\dot{\tilde{q}}_f + \tilde{q}_f = S(t) \quad \text{where } \tilde{q}_f(t) = \tilde{q}(t)$$

$$\text{Also } \tilde{q}_f(0) = \dot{q}_f(0) = 0$$

& $\tilde{q}_f(0) = \dot{q}_f(0) = 1$ satisfy initial conditions for the causal Green's function

$$\therefore \tilde{q}(t) = \begin{cases} 0 & t < 0 \\ -e^{-t} + e^{-t}(1+t) & t > 0 \end{cases}$$

is the response causal response to a delta function at $t = 0$.

\therefore The Green's causal Green's function, which is the causal response to a δ delta function at t' is simply

$$G(t, t') = \begin{cases} 0 : t < t' \\ (t-t') e^{-(t-t')} : t > t' \end{cases}$$

\therefore Now for a general force $F(t')$,

$$q(t) = \int_{-\infty}^t G(t, t') F(t') dt'$$

The upper limit of integration is t because, the causal Green's function is such that motion is unaffected (namely $q=0$) before the application of the force at t' . In other words, to find the response at t , one only need consider the forces applied before t . Hence the limits of integration are from $-\infty$ to t .

3. PROBLEM 4.1 HF

Consider the EOM $\ddot{q} - q = 0$

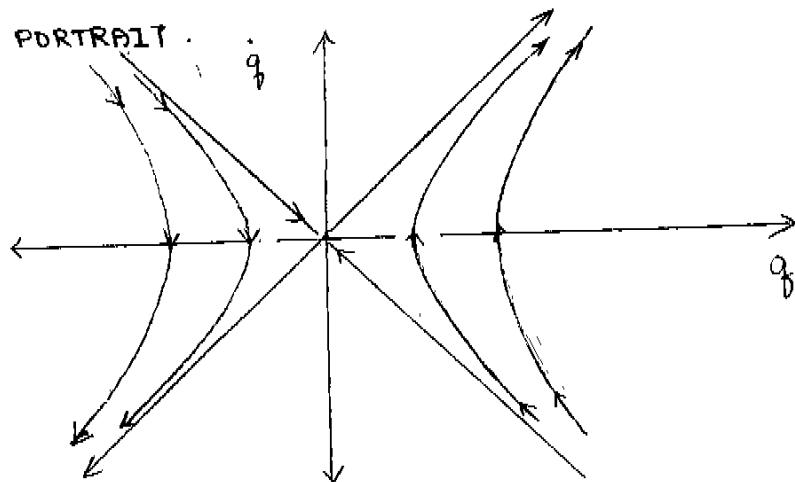
Characteristic eqn. $\lambda^2 - 1 = 0 \Rightarrow \lambda_{\pm} = \pm 1$

\therefore General soln. $q(t) = Ae^t + Be^{-t}$ for constants A, B
 $\dot{q}(t) = Ae^t - Be^{-t}$

$$\therefore \cancel{\dot{q}^2} - q^2 = -4AB \rightarrow ①$$

Eqn. ① gives the plot of a hyperbola on a \dot{q} vs q phase portrait.

∴ PHASE PORTRAIT



From previous chapters, we know Lagrangian for this system is merely $\mathcal{L} = \frac{1}{2} \dot{q}_1^2 + \frac{1}{2} \dot{q}_2^2$.

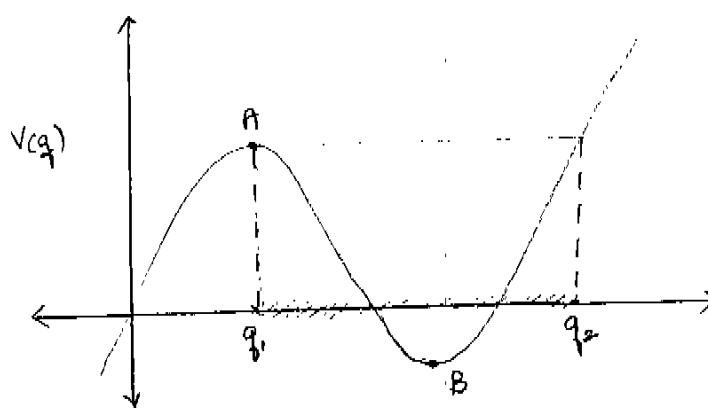
∴ $H = \dot{q}_1 \frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \mathcal{L}$ is conserved since $\frac{\partial H}{\partial t} = 0$

$$\therefore H = \frac{1}{2} \dot{q}_1^2 - \frac{1}{2} q_1^2 = \text{constant}$$

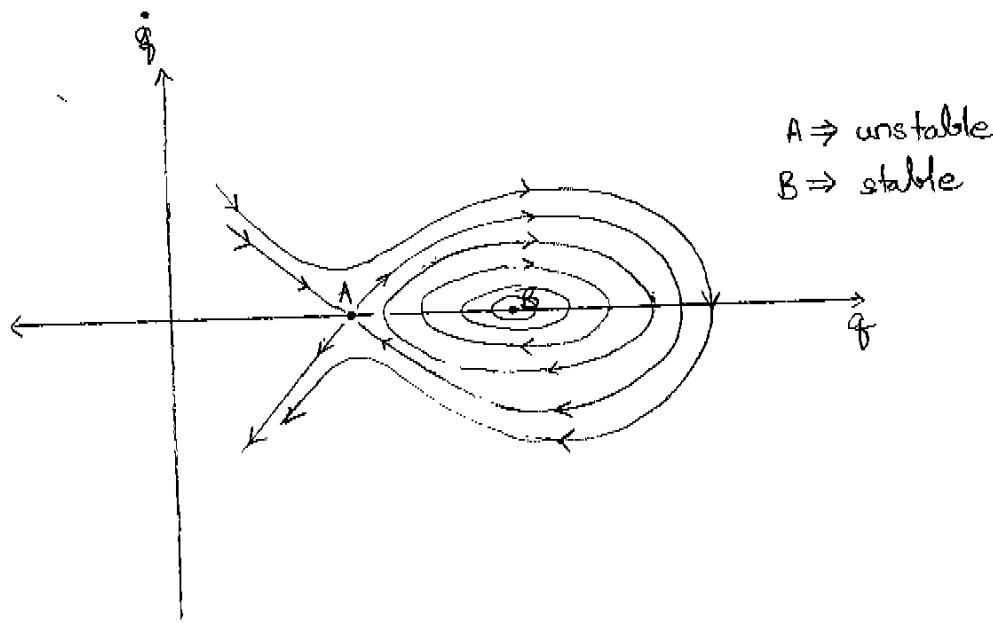
i.e. $\dot{q}_1^2 - q_1^2 = \text{constant}$, the Implicit Eqn. of our Phase Portrait!!!

Note that in this case Energy = H and hence Energy is also conserved.

b)

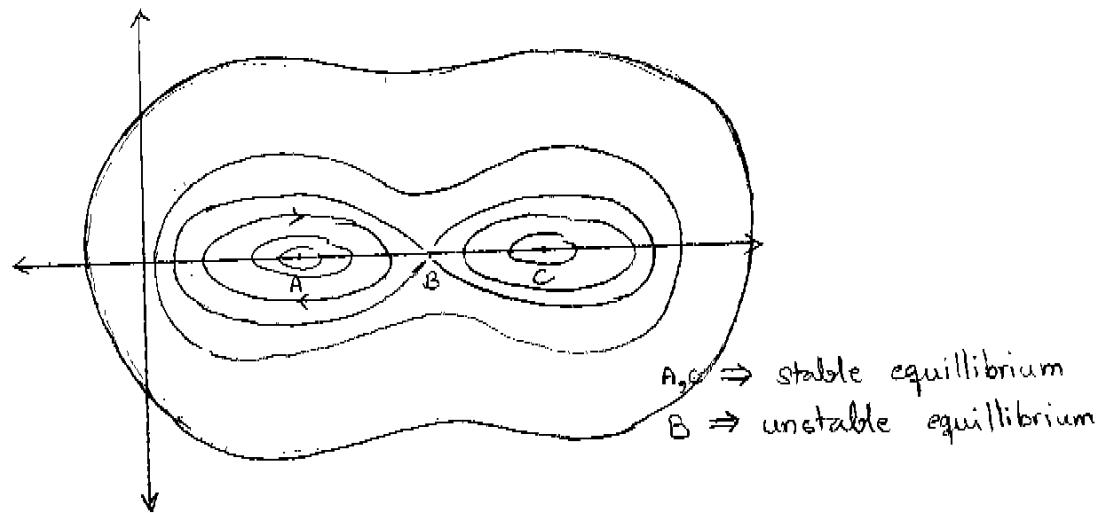


For $q_1 \leq q_0 \leq q_2$, there exists some velocity \dot{q}_0 , so that with initial conditions (q_0, \dot{q}_0) bounded motion results. q_0 is shown by the hatched section.



4. PROBLEM 4.2 HF

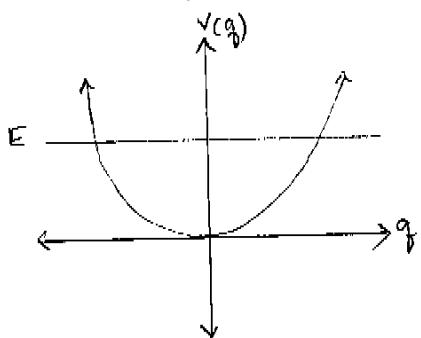
For the double well potential, we have pts. of stable and unstable equilibrium. Also, since $V(q) \rightarrow \infty$ for $q \rightarrow \pm \infty$, clearly only bounded motion is possible for any given finite initial energy.



5. PROBLEM 4.4 HF

For a Duffing oscillator, let
 $V(q) = \frac{q^2}{2} + \frac{\epsilon}{4} q^4, \epsilon > 0$

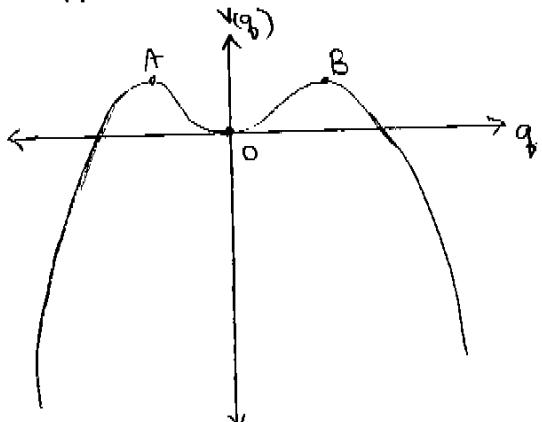
- a) For $\epsilon > 0$, it is clear that $V(q)$ is given by :



\therefore Since $V(q) \rightarrow +\infty$ for $q \rightarrow \pm\infty$, given any energy E , there will always be two turning pts. for the trajectory where $E = V(q)$. \therefore we only get bounded motion

This argument doesn't work for θ co-ordinate of a pendulum.
~~The θ co-ordinate itself is bounded, i.e. $0 < \theta < \pi$~~
~~where $-\pi \leq \theta \leq \pi$. Thus clearly one can only have bounded motion since $V(\theta)$ is bounded. Any E higher than V_{\max} results in unbounded motion.~~

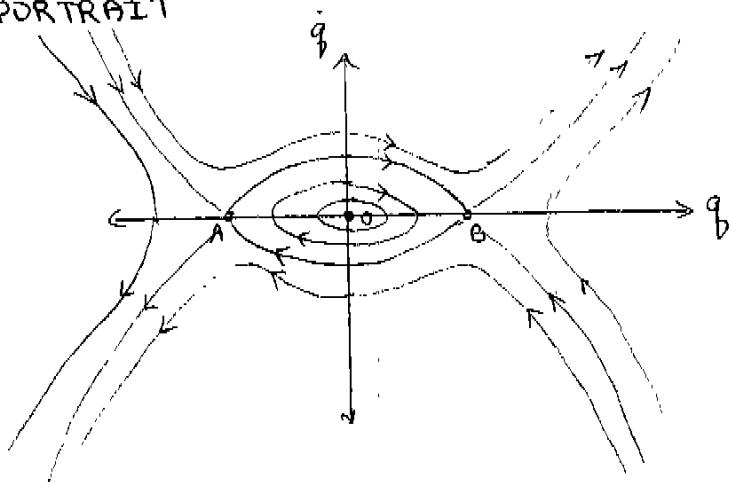
- b) Suppose $\epsilon < 0$. Then a plot of $V(q)$ is given by



This is essentially because for small q , the quadratic term dominates while for large q , the q^4 term dominate with its negative co-efficient.

A, B are unstable equilibrium pts.
 O is a stable equilibrium pt.

PHASE PORTRAIT



- c) For $V(q_f) = -\frac{q_f^2}{2} + \frac{\epsilon}{4} q_f^4$ one gets the double well potential of problem 4.2. Refer to solution of problem 4.